NAME: **STUDENT #:**

- There is a total of 43 marks; the maximum grade is 40 (3 bonus marks)
- Check that you have a total of 6 distinct pages and notify your TA if this is not the case
- Calculators are not allowed
- Phones and other devices should be turned off and hidden
- Have your student card face up on your desk
- You have to show all your work for all the questions, except the True/False and multiple choice

Question 1. You are given that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. Find the coordinate vector of $\begin{bmatrix} -9 \\ 13 \\ 22 \end{bmatrix}$ with respect to $\mathcal{B}$. You need to show some work; answers by observation will not be accepted. [2 marks]

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -9 \\ 0 & 1 & 1 & 13 \\ 3 & 0 & 1 & 22 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -9 \\ 0 & 1 & 1 & 13 \\ 0 & 0 & 7 & 49 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -9 \\ 0 & 1 & 1 & 13 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -9 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

So $\begin{bmatrix} -9 \\ 13 \\ 22 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

Question 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x \\ x-y \end{bmatrix}$.

a. Show that T is a linear transformation. [5 marks]

b. Find the standard matrix for T. [2 marks]

c. Consider $S : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ x+y \\ 0 \end{bmatrix}$. For each of $S \circ T$ and $T \circ S$, give an expression (formula), or explain why the composition does not exist. [3 marks]

a) Let $\vec{u}, \vec{v} \in \mathbb{R}^2$, $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.

$$T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} 5(u_1+v_1) \\ u_1+v_1-u_2-v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 5u_1+5v_1 \\ (u_1-u_2)+(v_1-v_2) \end{bmatrix} = \begin{bmatrix} 5u_1 \\ u_1-u_2 \end{bmatrix} + \begin{bmatrix} 5v_1 \\ v_1-v_2 \end{bmatrix} = T(\vec{u}) + T(\vec{v})$$

For $c \in \mathbb{R}$,

$$T(c\vec{u}) = T\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} 5cu_1 \\ cu_1 - cu_2 \end{bmatrix}$$

$$= c\begin{bmatrix} 5u_1 \\ u_1-u_2 \end{bmatrix} = c\,T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = c\,T(\vec{u}).$$

We conclude that T is a linear transformation

b) $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ so the standard matrix for T is $\begin{bmatrix} 5 & 0 \\ 1 & -1 \end{bmatrix}$.

c) $$(S \circ T)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = S\left(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = S\left(\begin{bmatrix} 5x \\ x-y \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2(5x) \\ (5x)+(x-y) \\ 0 \end{bmatrix} = \begin{bmatrix} 10x \\ 6x-y \\ 0 \end{bmatrix}$$

$T \circ S$ is not defined because the codomain of S is different than the domain of T.

Question 3. Let $A = \begin{array}{c} \text{col} \\ \begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 1 & 1 & 2 & 6 & 8 \\ 3 & 2 & 6 & 14 & 19 \end{bmatrix} \end{array}$. You are given that the RREF of A is $R = \begin{bmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ row

a. Find bases for $\mathrm{Col}(A)$, $\mathrm{Row}(A)$, and $\mathrm{Null}(A)$. [5 marks]

b. Give $\mathrm{rank}(A)$ and nullity(A). [2 marks]

c. State the rank-nullity theorem and confirm that it holds for A. [2 marks]

a) From the leading entries in R we see that:

- a basis for $\mathrm{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$
- a basis for $\mathrm{Row}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \\ 5 \end{bmatrix} \right\}$.

For the null(A), we solve

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 2 & 3 & 0 \\ 0 & 1 & 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} x_1 &= -2s - 2t - 3r \\ x_2 &= -4t - 5r \\ x_3 &= s \\ x_4 &= t \\ x_5 &= r \end{aligned}, \quad s, r, t \in \mathbb{R}$$

$x_1 \; x_2 \; \underbrace{x_3 \; x_4 \; x_5}_{\text{free}}$

$x_3 = s$
$x_4 = t$
$x_5 = r$

So Null(A) has basis $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

b) Rank(A) = 2 , nullity(A) = 3

c) Rank(A) + nullity(A) = # of columns

Here: 2 + 3 = 5

Question 4. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

a. Find the eigenvalues of A, and all their corresponding eigenvectors. [6 marks]

b. Is A diagonalizable? If yes, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. [2 marks]

a) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda)-10 = \lambda^2-5\lambda-6$

$\lambda = \frac{5 \pm \sqrt{25+4\cdot 6}}{2} = \frac{5\pm 7}{2} \begin{cases} 6 \\ -1 \end{cases}$

So the eigenvalues are: $\lambda_1 = 6$, $\lambda_2 = -1$.

For $\lambda_1 = 6$

$[A-6I \mid \vec{0}] = \left[\begin{array}{cc|c} -5 & 2 & 0 \\ 5 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -2/5 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2/5 \\ 1 \end{pmatrix} s,\ s\in\mathbb{R}$

So the ~~eigenvalues~~ eigenvectors for λ_1 are $\begin{bmatrix} 2/5 \\ 1 \end{bmatrix} s$, $s \in \mathbb{R}$.

For $\lambda_2 = -1$

$[A+I \mid \vec{0}] = \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 5 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t,\ t \in \mathbb{R}$

So the eigenvectors for λ_2 are $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t$, $t \in \mathbb{R}$.

b) A is diagonalizable, since it has 2 linearly independent eigenvectors, for example $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ for $\lambda_1 = 6$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for $\lambda_2 = -1$. So $A = PDP^{-1}$ with

$P = \begin{bmatrix} 2 & -1 \\ 5 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$.

True/False and multiple choice questions

Question 5. Circle T for *True* and F for *False*. Do not justify your answers, just circle your choice. [1 mark *each*]

T F Let T, S be linear transformations. If both $T \circ S$ and $S \circ T$ are defined, then they are equal.

T F If the characteristic polynomial of a 3×3 matrix has 3 distinct roots, then this matrix must be diagonalizable.

T F The zero vector is an eigenvector of every matrix.

T F For every linear transformation T, we have that $\mathbf{0} \in \ker T$.

T F If A is a square matrix and the system $A\mathbf{x} = \mathbf{0}$ has a nonzero solution, then the columns of A must be linearly dependent.

Question 6. In the following do not justify your answer, just circle one letter. [2 marks *each*]

- Only **one** of the following is an eigenvector for $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$. Which one is it?

A $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ **D** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **E** $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- For which k is the following set linearly independent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ k \end{bmatrix} \right\}$$

$\begin{matrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & k \end{matrix} \to \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & k-1 \end{matrix}$

A Only for $k = 1$ **B** For all $k \in \mathbb{R}$ with $k \neq 1$ **C** For no k at all

D For all $k \in \mathbb{R}$ **E** For all $k \in \mathbb{R}$ with $k \neq 0$

$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

- Let $T : \mathbb{R}^2 \to M_{2\times 2}$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$?

A $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}$ **C** $\begin{bmatrix} -2 & 0 \\ -3 & -2 \end{bmatrix}$ **D** $\begin{bmatrix} 2 & 0 \\ -1 & -4 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

Question 7 (Bonus question). Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

For any eigenvalue of this matrix, what is the dimension of the corresponding eigenspace? Explain briefly. (Hint: this can be answered with hardly any calculations, or by observation.) [3 marks]

See question 7 in version B

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Question 1. You are given that $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$. Find the coordinate vector of $\begin{bmatrix} -8 \\ 16 \\ 11 \end{bmatrix}$ with respect to $\mathcal{B}$. You need to show some work; answers by observation will not be accepted. [2 marks]

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -8 \\ 0 & 1 & 1 & 16 \\ 3 & 0 & 1 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -8 \\ 0 & 1 & 1 & 16 \\ 0 & 0 & 7 & 35 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -8 \\ 0 & 1 & 1 & 16 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -8 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

So $\begin{bmatrix} -8 \\ 16 \\ 11 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 11 \\ 5 \end{bmatrix}$

Question 2. Let $A = \begin{array}{c} \begin{array}{ccccc} \text{col} & & & & \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 2 & 4 \\ 2 & 3 & 8 & 4 & 29 \\ 1 & 1 & 3 & 2 & 11 \end{bmatrix} \end{array}$. You are given that the RREF of A is $R = \begin{bmatrix} 1 & 0 & 1 & 2 & 4 \\ 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.) row

a. Find bases for $\mathrm{Col}(A)$, $\mathrm{Row}(A)$, and $\mathrm{Null}(A)$. [5 marks]

b. Give $\mathrm{rank}(A)$ and $\mathrm{nullity}(A)$. [2 marks]

c. State the rank-nullity theorem and confirm that it holds for A. [2 marks]

a) From the leading entries in R we see that:

- a basis for $\mathrm{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$
- a basis for $\mathrm{Row}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 7 \end{bmatrix} \right\}$.

For $\mathrm{Null}(A)$, we have

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 4 & 0 \\ 0 & 1 & 2 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline x_1 & x_2 & x_3 & x_4 & x_5 & \end{array}\right] \Rightarrow \begin{aligned} x_1 &= -s - 2t - 4r \\ x_2 &= -2s \qquad -7r \\ x_3 &= s \\ x_4 &= t \\ x_5 &= r \end{aligned}$$

$\underbrace{x_1 \; x_2}_{\text{basic}}$ $\underbrace{x_3 \; x_4 \; x_5}_{\text{free}}$

$x_3 = s$
$x_4 = t$
$x_5 = r$

So $\mathrm{Null}(A)$ has basis $\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -7 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

b), c) Same as question 3, version A

Question 3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x \\ x - y \end{bmatrix}$.

a. Show that T is a linear transformation. [5 marks]

b. Find the standard matrix for T. [2 marks]

c. Consider $S : \mathbb{R}^2 \to \mathbb{R}^3$ given by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ x + y \\ 0 \end{bmatrix}$. For each of $S \circ T$ and $T \circ S$, give an expression (formula), or explain why the composition does not exist. [3 marks]

Same as question 2 in version A

Question 4. Let $A = \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}$.

a. Find the eigenvalues of A, and all their corresponding eigenvectors. [6 marks]

b. Is A diagonalizable? If yes, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. [2 marks]

a) $\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 5 \\ 4 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)-20$

$= \lambda^2 - 3\lambda - 18$

$\lambda = \frac{3 \pm \sqrt{9+4\cdot 18}}{2} = \frac{3 \pm \sqrt{81}}{2}$

$= \frac{3 \pm 9}{2} = \begin{cases} 6 \\ -3 \end{cases}$

So the eigenvalues are $\lambda_1 = 6$, $\lambda_2 = -3$

For $\lambda_1 = 6$

$[A-6I \mid \vec{0}] = \left[\begin{array}{cc|c} -5 & 5 & 0 \\ 4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s$, $s \in \mathbb{R}$

So the eigenvectors for $\lambda_1 = 6$ are $\begin{bmatrix} 1 \\ 1 \end{bmatrix} s$, $s \in \mathbb{R}$

For $\lambda_2 = -3$

$[A+3I \mid \vec{0}] = \left[\begin{array}{cc|c} 4 & 5 & 0 \\ 4 & 5 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 5/4 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/4 \\ 1 \end{bmatrix} t$, $t \in \mathbb{R}$

So the eigenvectors are $\begin{bmatrix} -5/4 \\ 1 \end{bmatrix} t$, $t \in \mathbb{R}$.

b) A is diagonalizable, since it has 2 linearly independent eigenvectors, for example $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda_1 = 6$, and $\begin{bmatrix} -5 \\ 4 \end{bmatrix}$ for $\lambda_2 = -3$.

So $A = PDP^{-1}$ with

$P = \begin{bmatrix} 1 & -5 \\ 1 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 6 & 0 \\ 0 & -3 \end{bmatrix}$

True/False and multiple choice questions

Question 5. Circle T for *True* and F for *False.* Do not justify your answers, just circle your choice. [1 mark *each*]

T F For every linear transformation T, the zero vector belongs in the kernel of T.

T F If the characteristic polynomial of a 3×3 matrix has 3 distinct roots, then this matrix must be diagonalizable.

T F The zero vector is an eigenvector of every matrix.

T F If A is a square matrix and the system $A\mathbf{x} = \mathbf{0}$ has a unique solution, then the columns of A must be linearly dependent.

T F Let T, S be linear transformations. If both $T \circ S$ and $S \circ T$ are defined, they are equal.

Question 6. In the following do not justify your answer, just circle one letter. [2 marks *each*]

- Only **one** of the following is an eigenvector for $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$. Which one is it?

A $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ **B** $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ **C** $\begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$ **D** $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ **E** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

- For which k is the following set linearly independent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ k \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & k & 0 \end{matrix} \rightarrow \begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & k-1 & 0 \end{matrix} \rightarrow \begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & k-1 & 1 \end{matrix}$$

A For all $k \in \mathbb{R}$ **B** For all $k \in \mathbb{R}$ with $k \neq 0$ **C** For no k at all

D Only for $k = 1$ **E** For all $k \in \mathbb{R}$ with $k \neq 1$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Let $T : \mathbb{R}^2 \to M_{2\times 2}$ be a linear transformation such that $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}$. What is $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right)$?

A $\begin{bmatrix} -2 & 0 \\ -3 & -2 \end{bmatrix}$ **B** $\begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}$ **C** $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$ **D** $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & 0 \\ -1 & -4 \end{bmatrix}$

Question 7 (Bonus question). Let

$$A = \begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \\ 0 & 5 & 4 & 3 & 2 & 1 \\ 0 & 0 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For any eigenvalue of this matrix, what is the dimension of the corresponding eigenspace? Explain briefly. (Hint: this can be answered with hardly any calculations, or by observation.) [3 marks]

The dimension is 1 for every eigenvalue.

This is because it has 6 distinct eigenvalues (the diagonal entries), so each one has an eigenspace of dimension 1.

OR

This is because it has 6 distinct eigenvalues (the diagonal entries), so ~~each one~~ the rank of $(A - \lambda I)$ is always 5, for every eigenvalue λ, and thus the eigenspace of λ, given by $[A - \lambda I \mid \vec{0}]$ has dimension $6 - 5 = 1$, from the rank-nullity theorem